Midsemestral examination
Second semester 2011
M.Math.IInd year
Algebraic Number Theory
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Q 1.
If L/K is a Galois extension of number fields and P is a prime ideal of O_K, then prove that the Galois group acts transitively on the set of prime ideals of O_L lying over P.

OR

If a Dedekind domain is a UFD, show that it must be a PID.

Q 2.
Show that the class number of $\mathbf{Q}(\sqrt{-5})$ is 2.

OR

Consider the field $K = \mathbf{Q}(\alpha)$ where $f(\alpha) = 0$ with $f(X) = X^3 - X - 1$. Determine the discriminant and deduce that $O_K = \mathbf{Z}[\alpha]$. Find also the best integer given by the Minkowski bound.

Q 3.
Let L/K be an extension of number fields and let $\mathcal{D}$ be the discriminant ideal. If a prime ideal P of O_K contains $\mathcal{D}$, prove that P ramifies in L.

OR

Let A be a Dedekind domain and K, its quotient field. Let L be a Galois extension of K and B, the integral closure of A in L. If I is a proper ideal of A, prove that $IB \cap A = I$.

Q 4.
Consider the cyclotomic field $\mathbf{Q}(\zeta_n)$ generated by n-th roots of unity. If p is a prime number not dividing n, describe the ideals $P_1, \cdots, P_g$ of $\mathbf{Z}[\zeta_n]$ lying over p. Find also the norm of each ideal P_i.

OR

Describe with proof the totally ramified extensions of $\mathbf{Q}_p$.

Q 5.
Let l/k be an unramified extension of p-adic fields (that is, extensions of $\mathbf{Q}_p$). Prove that the norm map from the units of l to those of k is surjective.

OR

If V is a finite-dimensional vector apace over $\mathbf{Q}_p$, show that there is a unique extension of the p-adic absolute value $|.|_p$ to V.

Q 6.
Let $p \equiv 1 \bmod 4$ be a prime. Prove that $x^2 - py^2 = -1$ has infinitely many solutions.

OR

Prove that the fundamental unit of $\mathbf{Q}(\sqrt{d^2-1})$ is $d+\sqrt{d^2-1}$ for each square-free integer $d \geq 2$.